IRREDUCIBLE REPRESENTATIONS OF THE CPT GROUPS IN QED

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Abstract: We construct the inequivalent irreducible representations (IIR's) of the CPT groups of the Dirac field operator $\hat{\psi}$ and the electromagnetic quantum potential $\hat{A}_\mu$. The results are valid both for free and interacting (QED) fields. Also, and for the sake of completeness, we construct the IIR's of the CPT group of the Dirac equation.

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1 Introduction

The CPT group $G_{\hat{\Theta}}(\hat{\psi})$ of the Dirac quantum field $\hat{\psi}$ is isomorphic to the direct product of the quaternion group Q and the cyclic group of two elements $\mathbb{Z}_2$ ([1] and [2]):

$$G_{\hat{\Theta}}(\hat{\psi}) \cong Q \times \mathbb{Z}_2. \tag{1}$$

$\hat{\Theta}$ is the product $\hat{C} * \hat{P} * \hat{T}$ of the three operators: $\hat{C}$ (charge conjugation), $\hat{P}$ (space inversion), $\hat{T}$ (time reversal); the operation $\hat{A} * \hat{B}$, where $\hat{A}$ and $\hat{B}$ are any of the operators $\hat{C}$, $\hat{P}$, $\hat{T}$, is given by $(\hat{A} * \hat{B}) \cdot \hat{\psi} = (\hat{A}\hat{B})^\dagger \hat{\psi} (\hat{A}\hat{B})$. $G_{\hat{\Theta}}(\hat{\psi})$ is one of the nine non abelian groups of a total of fourteen groups with sixteen elements [3]; only three of them have three generators.

The isomorphism $G_{\hat{\Theta}}(\hat{\psi}) \to Q \times \mathbb{Z}_2$ is given by the relations (2):

$$\begin{aligned}
1 &\mapsto (1,1) & -1 &\mapsto (-1,1) \\
\hat{C} &\mapsto (1,-1) & -\hat{C} &\mapsto (-1,-1) \\
\hat{P} &\mapsto (\iota,1) & -\hat{P} &\mapsto (-\iota,1) \\
\hat{T} &\mapsto (\gamma,1) & -\hat{T} &\mapsto (-\gamma,1) \\
\hat{C} * \hat{P} &\mapsto (\iota,-1) & -\hat{C} * \hat{P} &\mapsto (-\iota,-1) \\
\hat{C} * \hat{T} &\mapsto (\gamma,-1) & -\hat{C} * \hat{T} &\mapsto (-\gamma,-1) \\
\hat{P} * \hat{T} &\mapsto (\kappa,1) & -\hat{P} * \hat{T} &\mapsto (-\kappa,1) \\
\hat{\Theta} &\mapsto (\kappa,-1) & -\hat{\Theta} &\mapsto (-\kappa,-1);
\end{aligned} \tag{2}$$

where ι, γ, κ are the three imaginary units defining the quaternion numbers.

On the other hand, the action of the $\hat{C}$, $\hat{P}$ and $\hat{T}$ operators on the Maxwell electromagnetic 4-potential $\hat{A}_\mu$ is given by (3) ([4] and [5]), with the Minkowski space-time metric $diag(1,-1,-1,-1)$:

$$\begin{aligned}
\hat{P}\hat{A}^\mu(\mathbf{x},t)\hat{P}^{-1} &= \hat{A}_\mu(-\mathbf{x},t), \\
\hat{C}\hat{A}^\mu(\mathbf{x},t)\hat{C}^{-1} &= -\hat{A}^\mu(\mathbf{x},t), \\
\hat{T}\hat{A}^\mu(\mathbf{x},t)\hat{T}^{-1} &= \hat{A}_\mu(\mathbf{x},-t).
\end{aligned} \tag{3}$$

This leads to the CPT group of the electromagnetic field operator, $G_{\hat{\Theta}}(\hat{A}_\mu)$, which is an abelian group of eight elements with three generators, isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \mathbb{Z}_2^3$. Geometrically, $\mathbb{Z}_2^3$ is isomorphic to the group D_{2h}, the symmetry group of the parallelepiped.

The isomorphism

$$G_{\hat{\Theta}}(\hat{A}_\mu) \to \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \tag{4}$$

is given by

$$\begin{aligned}
1 &\mapsto (e_1, e_2, e_3) \\
\hat{C} &\mapsto (a_1, e_2, e_3) \\
\hat{P} &\mapsto (e_1, a_2, e_3) \\
\hat{T} &\mapsto (e_1, a_2, a_3) \\
\hat{P} * \hat{T} &\mapsto (e_1, a_2, a_3) \\
\hat{C} * \hat{P} &\mapsto (a_1, a_2, e_3) \\
\hat{C} * \hat{T} &\mapsto (a_1, e_2, a_3) \\
\hat{\Theta} &\mapsto (a_1, a_2, a_3);
\end{aligned} \tag{5}$$

where e_i and a_i (for $i = 1, 2, 3$) represent, respectively, the identity and the "minus 1" of each of the $\mathbb{Z}_2$ groups.

As the CPT transformation properties of the interacting $\hat{\psi} - \hat{A}_\mu$ fields are the same as for the free fields ([5] and [6]), it is then clear that the complete CPT group for QED, $G_{\hat{\Theta}}(QED)$, is the direct product of the two above mentioned groups, $G_{\hat{\Theta}}(\hat{\psi})$ and $G_{\hat{\Theta}}(\hat{A}_\mu)$, i.e.,

$$G_{\hat{\Theta}}(QED) = G_{\hat{\Theta}}(\hat{\psi}) \times G_{\hat{\Theta}}(\hat{A}_\mu), \tag{6}$$

which turns out to be a group of order $|G_{\hat{\Theta}}(QED)| = 16 \times 8 = 128$.

Thus,

$$G_{\hat{\Theta}}(QED) \cong (Q \times \mathbb{Z}_2) \times \mathbb{Z}_2^3. \tag{7}$$

Since, as will be shown below, $G_{\hat{\Theta}}(\hat{\psi})$ has ten inequivalent irreducible representations (IIR's), while $G_{\hat{\Theta}}(\hat{A}_\mu)$ has only eight IIR's, then, the total number of IIR's of $G_{\hat{\Theta}}(QED)$ is eighty: sixty four of them 1-dimensional and sixteen 2-dimensional.

It is interesting at this point to make a comment about the geometrical content of $G_{\hat{\Theta}}(QED)$, which is a set of points in spheres. In fact, each factor $\mathbb{Z}_2$ is a 0-sphere S^0, while Q can be thought as a subset of eight points in the 3-sphere S^3. Thus, topologically

$$G_{\hat{\Theta}}(QED) \subset SU(2) \times (U(1))^4. \tag{8}$$

In section 2 we construct by tensor products the ten IIR's and character tables of $G_{\hat{\Theta}}(\hat{\psi})$. In section 3 we construct the eight IIR's of $G_{\hat{\Theta}}(\hat{A}_\mu)$; in this case, since all representations are 1-dimensional, the characters coincide with the representations. In

section 4 we construct the IIR's of the CPT group corresponding to the Dirac equation for the wave function ψ, $G_{\Theta}^{(2)}(\psi)$ in [1], and compare the result with the case for the Dirac field operator $\hat{\psi}$. Finally, in section 5, we discuss the relation between the ¨fermionic" and ¨bosonic" CPT groups and comment about the relation of these groups with some approaches to quantum paradoxes.

2 IIR's of $G_{\hat{\Theta}}(\hat{\psi})$

The irreducible representations (irrep's) of the direct product of two groups, $G \times H$, are the tensor products of the irrep's of each of the factors, namely the set $\{r_G \otimes r_H\}$ [1] for all $r'_G s$, irrep's of G and all $r'_H s$, irrep's of H [7]. If G is the quaternion group, which, as is well known, has five IIR's φ_μ (with $\varphi_1, ..., \varphi_4$ 1-dimensional and φ_5 2-dimensional)[8] and characters table given by table 1; and H is $\mathbb{Z}_2$ (with IIR's ψ_1 and ψ_2), then $G_{\hat{\Theta}}(\hat{\psi})$ has ten IIR's:

$$\phi_\alpha = \varphi_\alpha \otimes \psi_1, \quad \phi_{\alpha+4} = \varphi_\alpha \otimes \psi_2, \quad \alpha = 1,2,3,4, \tag{9}$$

$$\phi_9 = \varphi_5 \otimes \psi_1, \quad \phi_{10} = \varphi_5 \otimes \psi_2. \tag{10}$$

$\phi_1, ..., \phi_8$ are 1-dimensional, while ϕ_9 and ϕ_{10} are 2-dimensional.

Ch Q	$[1]$	$[-1]$	$2[\iota]$	$2[\gamma]$	$2[\kappa]$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Table 1: Characters table of Q.

The corresponding characters are the functions

$$\kappa_\alpha = \chi_\alpha \varphi_1 : Q \times \mathbb{Z}_2 \to \mathbb{C}, \quad \kappa_\alpha(q,h) = \chi_\alpha(q)\varphi_1(h), \tag{11}$$

$$\kappa_{\alpha+4} = \chi_\alpha \varphi_2 : Q \times \mathbb{Z}_2 \to \mathbb{C}, \quad \kappa_{\alpha+4}(q,h) = \chi_\alpha(q)\varphi_2(h), \tag{12}$$

$$\kappa_9 = \chi_5 \varphi_1 : Q \times \mathbb{Z}_2 \to \mathbb{C}, \quad \kappa_9(q,h) = \chi_5(q)\varphi_1(h), \tag{13}$$

$$\kappa_{10} = \chi_5 \varphi_2 : Q \times \mathbb{Z}_2 \to \mathbb{C}, \quad \kappa_{10}(q,h) = \chi_5(q)\varphi_2(h). \tag{14}$$

[1] $\otimes = \otimes_{\mathbb{C}}$ is the tensor product over the complex numbers

These representations and characters (for conjugate classes) are explicitly given by the tables 2 and 3, respectively, where $\lambda_i = tr\phi_i$, and the conjugate classes are given by the relations (15):

$$\begin{aligned}
[\hat{I}] &= [(1,e)] = \{(1,e)\}, \\
[\hat{C}] &= [(1,a)] = \{(1,a)\}, \\
[-\hat{I}] &= [(-1,e)] = \{(-1,e)\}, \\
[-\hat{C}] &= [(-1,a)] = \{(-1,a)\}, \\
[\hat{P}] &= [(\iota,e)] = \{(\iota,e),(-\iota,e)\}, \\
[\hat{C}*\hat{P}] &= [(\iota,a)] = \{(\iota,a),(-\iota,a)\}, \\
[\hat{T}] &= [(\gamma,e)] = \{(\gamma,e),(-\gamma,e)\}, \\
[\hat{C}*\hat{T}] &= [(\gamma,a)] = \{(\gamma,a),(-\gamma,a)\}, \\
[\hat{P}*\hat{T}] &= [(\kappa,e)] = \{(\kappa,e),(-\kappa,e)\}, \\
[\hat{\Theta}] &= [(\kappa,a)] = \{(\kappa,a),(-\kappa,a)\}; \qquad (15)
\end{aligned}$$

where e and a are the elements of $\mathbb{Z}_2$.

The character table of $G_{\hat{\Theta}}(\hat{\psi})$ is the same as the character table of D_{4h}, the group of symmetries of a prism with square base, though $Q \times \mathbb{Z}_2 \ncong D_{4h}$.

3 IIR's of $\mathbf{G}_{\hat{\Theta}}(\hat{\mathbf{A}}_{\mu})$

Since $G_{\hat{\Theta}}(\hat{A}_\mu) \cong \mathbb{Z}_2^3$ is an abelian group, all its irrep's are 1-dimensional. Since $|G_{\hat{\Theta}}(\hat{A}_\mu)| = 8$, then $G_{\hat{\Theta}}(\hat{A}_\mu)$ has eight IIR's, which can be identified with the corresponding characters. Using the isomorphism between $G_{\hat{\Theta}}(\hat{A}_\mu)$ and $\mathbb{Z}_2^3$ (eq. (5)), and the character table of $\mathbb{Z}_2$, we obtain the table of representations (ϕ_{ijk}) or characters (χ_{ijk}) for $G_{\hat{\Theta}}(\hat{A}_\mu)$, where $\phi_{ijk} = \chi_{ijk} = \psi_i\psi_j\psi_k$. See table 4.

4 IIR's of $\mathbf{G}^{(2)}_{\boldsymbol{\Theta}}(\boldsymbol{\psi})$

In [1] it was shown that at the level of the Dirac equation, for the 4-spinor ψ, exists two CPT groups, $G^{(1)}_{\Theta}(\psi)$ and $G^{(2)}_{\Theta}(\psi)$, whose elements are constructed with products of Dirac γ-matrices. Only $G^{(2)}_{\Theta}(\psi)$, which turns out isomorphic to the semi-direct product $D_4 \rtimes \mathbb{Z}_2$, where $D_4 \equiv DH_8$ is the dihedral group of eight elements (the group of symmetries of the square), is compatible with $G_{\hat{\Theta}}(\hat{\psi})$.

It is then of interest to construct the IIR's of $G_{\Theta}^{(2)}(\psi)$, both for completness and also for comparison with the IIR's of the operator group.

The action of $\mathbb{Z}_2 \cong \{1, -1\}$ on D_4 as a subgroup of S_4 (the symmetric group of 4 elements) is given by

$$\begin{aligned}
\lambda : \mathbb{Z}_2 \to Aut(D_4),& \\
1 \mapsto \lambda(1) &= Id_{D_4}, \\
\lambda(-1)(I) &= I, \\
\lambda(-1)(1234) &= (1234), \\
\lambda(-1)(24) &= (13), \\
\lambda(-1)(13) &= (24), \\
\lambda(-1)((12)(34)) &= (14)(23), \\
\lambda(-1)((14)(23)) &= (12)(34), \\
\lambda(-1)((13)(24)) &= (13)(24), \\
\lambda(-1)(1432) &= (1432).
\end{aligned} \tag{16}$$

Here

$$D_4 = \{I; (1234), (1432); (13)(24); (12)(34), (14)(23); (24), (13)\}, \tag{17}$$

where between semi-colons we have enclosed the five conjugation classes.

Then, the composition law in $D_4 \rtimes \mathbb{Z}_2$ is

$$(g', h')(g, h) = (g'\lambda(h')(g), h'h) : \tag{18}$$

$$(g', 1)(g, h) = (g'g, h) \tag{19}$$

$$(g', -1)(g, h) = (g'\lambda(-1)(g), -h), \tag{20}$$

where, as a set,

$$\begin{aligned}
D_4 \times \mathbb{Z}_2 = \{&(I, 1), (I, -1), (-I, 1), (-I, -1), (P, 1), (P, -1), (-P, 1), \\
&(-P, -1), (CT, 1), (CT, -1), (-CT, 1), (-CT, -1), (\Theta, 1), \\
&(\Theta, -1), (-\Theta, 1), (-\Theta, -1)\},
\end{aligned} \tag{21}$$

with inverses

$$(g, h)^{-1} = (\lambda(h)(g^{-1}), h). \tag{22}$$

From eq. (55) in [1], there is the isomorphism

$$D_4 \to \{I, -I, P, -P, CT, -CT, \Theta, -\Theta\} \tag{23}$$

given by

$$\begin{aligned} I &\mapsto I, \\ (1234) &\mapsto P, \\ (1432) &\mapsto -P, \\ (13)(24) &\mapsto -I, \\ (12)(34) &\mapsto \Theta, \\ (14)(23) &\mapsto -\Theta, \\ (24) &\mapsto -CT, \\ (13) &\mapsto CT \end{aligned} \tag{24}$$

while from eq. (60) in [1] and eq. (17), the isomorphism between $G_\Theta^{(2)}$ and $D_4 \rtimes \mathbb{Z}_2$,

$$G_\Theta^{(2)} \to D_4 \rtimes \mathbb{Z}_2, \tag{25}$$

is given by:

$$\begin{aligned} I &\mapsto (I, 1), & -I &\mapsto (-I, 1), \\ C &\mapsto (-\Theta, -1), & -C &\mapsto (\Theta, -1), \\ P &\mapsto (P, 1), & -P &\mapsto (-P, 1), \\ T &\mapsto (P, -1), & -T &\mapsto (-P, -1), \\ CP &\mapsto (CT, -1), & -CP &\mapsto (-CT, -1), \\ CT &\mapsto (CT, 1), & -CT &\mapsto (-CT, 1), \\ PT &\mapsto (-I, -1), & -PT &\mapsto (I, -1), \\ \Theta &\mapsto (\Theta, 1), & -\Theta &\mapsto (-\Theta, 1). \end{aligned} \tag{26}$$

It is easy to verify that $G_\Theta^{(2)}$ has ten conjugation classes, namely

$$\begin{aligned}
[I] &= = \{I\}, \\
[-I] &= = \{-I\}, \\
[C] &= = \{C, -C\}, \\
[T] &= = \{T\}, \\
[-T] &= = \{-T\}, \\
[P] &= = \{P, -P\}, \\
[CP] &= = \{CP, -CP\}, \\
[CT] &= = \{CT, -CT\}, \\
[PT] &= = \{PT, -PT\}, \\
[\Theta] &= = \{\Theta, -\Theta\}.
\end{aligned} \tag{27}$$

Then, being a finite group, $G_\Theta^{(2)}$ has as many IIR's as conjugation classes. Since the sum of the squares of the dimensions of these representations must be 10, a simple calculation leads to the existence of eight 1-dimensional irrep's φ_k, $k = 1, ...8$, and two 2-dimensional irrep's, φ_9 and φ_{10}.

φ_1 is the trivial representation $g \mapsto 1$, for all $g \in G_\Theta^{(2)}$. The three 1-dimensional irrep's φ_2, φ_3 and φ_4 are obtained taking into account:

1. D_4, $C_4 \times \mathbb{Z}_2$ and Q are invariant subgroups of $D_4 \rtimes \mathbb{Z}_2$.

2. The well known theorem by which if H is an invariant subgroup of G and $\varphi : G/H \to K < GL(V)$ is a representation of G/H over the vector space V, then $\varphi \circ p$ is a degenerate representation of G, where $p : G \to G/H$ is the canonical projection $p(g) =< g >= gH$ [9]. Thus, we have the commutative diagram:

$$\begin{array}{ccc} & \mathrm{K} & \\ \varphi \circ p \nearrow & & \nwarrow \varphi \\ \mathrm{G} & \xrightarrow[p]{} & G/H \end{array} \quad . \tag{28}$$

D_4 is given by the r.h.s. of eq. (23) and $C_4 =< \{P\} >$, so

$$C_4 \times \mathbb{Z}_2 = \{(I,1),(I,-1),(-I,1),(-I,-1),(P,1),(P,-1),(-P,1),(-P,-1)\}, \tag{29}$$

and

$$Q \cong < \{C, P\} >= \{I, C, P, CP, -I - C, -P, -CP\}. \tag{30}$$

In diagram (28), we choose $G = D_4 \rtimes \mathbb{Z}_2$ and H equal to D_4, $C_4 \times \mathbb{Z}_2$ and Q, respectively. Then

$$\frac{D_4 \rtimes \mathbb{Z}_2}{H} \cong \mathbb{Z}_2 = \{e, a\} \tag{31}$$

with

$$\begin{aligned}
e &= D_4, \qquad a = \{C, T, CP, PT, -C, -T, -CP, -PT\}, \\
e &= C_4 \times \mathbb{Z}_2, \qquad a = \{C, CP, CT, \Theta, -C, -CP, -CT, -\Theta\}, \\
e &= Q, \qquad a = \{T, CT, PT, \Theta, -T, -CT, -PT, -\Theta\},
\end{aligned} \tag{32}$$

respectively. The elements in the e's are represented by 1, while the elements in the a's are represented by -1 (see table 5).

As is well known, D_4 has five IIR's (one 2-dimensional and four 1-dimensional). The 2-dimensional irrep is given by

$$\begin{aligned}
I &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_{2\times 2} \equiv I \\
(13)(24) &\mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1_{2\times 2} \equiv -I \\
(1234) &\mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2 \equiv P \\
(1432) &\mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2 \equiv -P \\
(12)(34) &\mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\sigma_3 \equiv \Theta \\
(14)(23) &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3 \equiv -\Theta \\
(24) &\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 \equiv -CT \\
(13) &\mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -\sigma_1 \equiv CT,
\end{aligned} \tag{33}$$

where in the last identification we used the relations (24).

With the choice $C = i\sigma_1$, we obtain,

$$T = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} = iI, \quad CP = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3, \quad PT = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2, \tag{34}$$

which completes the 2-dimensional irrep φ_9 of $G_{\Theta}^{(2)}$; while with the choice $C=-i\sigma_1$, we obtain,

$$T=\begin{pmatrix}-i & 0\\ 0 & -i\end{pmatrix}=-iI,\quad CP=\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}=-i\sigma_3,\quad PT=\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}=-\sigma_2, \tag{35}$$

which completes the 2-dimensional irrep φ_{10} of $G_{\Theta}^{(2)}$. It can be easily verified that φ_9 and φ_{10} are inequivalent, i.e., it does not exist a matrix S such that $SMS^{\dagger}=M'$ (or $SMS^{-1}=M'$) for $M\in\varphi_9$ and $M'\in\varphi_{10}$.

The remaining four 1-dimensional IIR's, φ_5, φ_6, φ_7 and φ_8, are obtained from the orthogonality between the columns of the characters table for conjugation classes (completeness relation). Then, the orthogonality between the rows of the complete characters table is verified.

The IIR's and characters for $G_{\Theta}^{(2)}(\psi)\cong D_4\rtimes\mathbb{Z}_2$ are summarized in tables 5 and 6. Comparing with the tables for $G_{\hat{\Theta}}(\hat{\psi})$, we see that the 1-dimensional irrep's coincide:

$$\begin{gathered}\phi_1=\varphi_1,\ \phi_2=\varphi_3,\ \phi_3=\varphi_4,\ \phi_4=\varphi_2,\\ \phi_5=\varphi_8,\ \phi_6=\varphi_6,\ \phi_7=\varphi_7,\ \phi_8=\varphi_5.\end{gathered} \tag{36}$$

However, ϕ_9 is not equivalent to either φ_9 or φ_{10} and the same holds for ϕ_{10}.

5 Final comments

It is of interest to ask the question if, from the point of view of group theory, the behaviour of the photon field $\hat{A}_\mu$ under the C,P,T transformation is independent of the behaviour of the electron-positron field $\hat{\psi}$. The answer to this question is afirmative. In fact, in $G_{\hat{\Theta}}(\hat{\psi})$ there is only one $\mathbb{Z}_2$ subgroup, namely that generated by $\hat{C}:\ \mathbb{Z}_2\cong\{\hat{I},\hat{C}\}$. Then, $G_{\hat{\Theta}}(\hat{A}_\mu)\cong\mathbb{Z}_2^3$ is not a subgroup of $G_{\hat{\Theta}}(\hat{\psi})$:

$$G_{\hat{\Theta}}(\hat{A}_\mu)\not< G_{\hat{\Theta}}(\hat{\psi}). \tag{37}$$

The same happens in relation to $G_{\Theta}^{(2)}(\psi)$. In this group there are three $\mathbb{Z}_2$-subgroups:

$$\mathbb{Z}_2^{(1)}=\{I,CT\},\ \mathbb{Z}_2^{(2)}=\{I,PT\},\ \mathbb{Z}_2^{(3)}=\{I,\Theta\}. \tag{38}$$

However, the table of $\mathbb{Z}_2^3$ has eight 1's, while the table of $G_{\Theta}^{(2)}(\psi)$ has only seven 1's. So

$$G_{\hat{\Theta}}(\hat{A}_\mu)\not< G_{\Theta}^{(2)}. \tag{39}$$

Finally, in relation with the posible relevance of the CPT group structures in other areas of physics, besides field theory, we call the attention on the recently found relation between the groups $G_{\hat{\Theta}}(\hat{\psi}) \cong Q \times \mathbb{Z}_2$, $G_{\Theta}^{(2)}(\psi) \cong D_4 \rtimes \mathbb{Z}_2$ and $G_{\hat{\Theta}}(\hat{A}_\mu) \cong \mathbb{Z}_2^3$, and fundamental theorems in the context of quantum paradoxes, like the Kochen-Specker theorem [10].

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IIR's	$\hat{I}$	$\hat{C}$	$-\hat{I}$	$-\hat{C}$	$\hat{P}$	$\hat{C}*\hat{P}$	$-\hat{P}$	$-\hat{C}*\hat{P}$	$\hat{T}$	$\hat{C}*\hat{T}$	$-\hat{T}$	$-\hat{C}*\hat{T}$	$\hat{P}*\hat{T}$	$\hat{\Theta}$	$-\hat{P}*\hat{T}$	$-\hat{\Theta}$
$G_{\hat{\Theta}}(\hat{\psi})$	$(1,e)$	$(1,a)$	$(-1,e)$	$(-1,a)$	(ι,e)	(ι,a)	$(-\iota,e)$	$(-\iota,a)$	(γ,e)	(γ,a)	$(-\gamma,e)$	$(-\gamma,a)$	(κ,e)	(κ,a)	$(-\kappa,e)$	$(-\kappa,a)$
ϕ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ϕ_2	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
ϕ_3	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
ϕ_4	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1
ϕ_5	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
ϕ_6	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1
ϕ_7	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1
ϕ_8	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1
ϕ_9	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}i & 0\\0 & -i\end{pmatrix}$	$\begin{pmatrix}i & 0\\0 & -i\end{pmatrix}$	$\begin{pmatrix}-i & 0\\0 & i\end{pmatrix}$	$\begin{pmatrix}-i & 0\\0 & i\end{pmatrix}$	$\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & i\\i & 0\end{pmatrix}$	$\begin{pmatrix}0 & i\\i & 0\end{pmatrix}$	$\begin{pmatrix}0 & -i\\-i & 0\end{pmatrix}$	$\begin{pmatrix}0 & -i\\-i & 0\end{pmatrix}$
ϕ_{10}	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}i & 0\\0 & -i\end{pmatrix}$	$\begin{pmatrix}-i & 0\\0 & i\end{pmatrix}$	$\begin{pmatrix}-i & 0\\0 & i\end{pmatrix}$	$\begin{pmatrix}i & 0\\0 & -i\end{pmatrix}$	$\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & i\\i & 0\end{pmatrix}$	$\begin{pmatrix}0 & -i\\-i & 0\end{pmatrix}$	$\begin{pmatrix}0 & -i\\-i & 0\end{pmatrix}$	$\begin{pmatrix}0 & i\\i & 0\end{pmatrix}$

Table 2: IIR's table of $G_{\hat{\Theta}}(\hat{\psi})$.

Ch $G_{\hat{\Theta}}(\hat{\psi})$	$[(1,e)]$	$[(1,a)]$	$[(-1,e)]$	$[(-1,a)]$	$2[(\iota,e)]$	$2[(\iota,a)]$	$2[(\gamma,e)]$	$2[(\gamma,a)]$	$2[(\kappa,e)]$	$2[(\kappa,a)]$
λ_1	1	1	1	1	1	1	1	1	1	1
λ_2	1	−1	1	−1	1	−1	1	−1	1	−1
λ_3	1	1	1	1	1	1	−1	−1	−1	−1
λ_4	1	−1	1	−1	1	−1	−1	1	−1	1
λ_5	1	1	1	1	−1	−1	1	1	−1	−1
λ_6	1	−1	1	−1	−1	1	1	−1	−1	1
λ_7	1	1	1	1	−1	−1	−1	−1	1	1
λ_8	1	−1	1	−1	−1	1	−1	1	1	−1
λ_9	2	2	−2	−2	0	0	0	0	0	0
λ_{10}	2	−2	−2	2	0	0	0	0	0	0

Table 3: Characters table of $G_{\hat{\Theta}}(\hat{\psi})$.

IIR's (Ch)	$\hat{I}$	$\hat{C}$	$\hat{P}$	$\hat{T}$	$\hat{P}*\hat{T}$	$\hat{C}*\hat{P}$	$\hat{C}*\hat{T}$	$\hat{\Theta}$
$G_{\hat{\Theta}}(\hat{A}_\mu)$	(e_1,e_2,e_3)	(a_1,e_2,e_3)	(e_1,a_2,e_3)	(e_1,e_2,a_3)	(e_1,a_2,a_3)	(a_1,a_2,e_3)	(a_1,e_2,a_3)	(a_1,a_2,a_3)
$\phi_{111}=\Phi_1$	1	1	1	1	1	1	1	1
$\phi_{211}=\Phi_2$	1	−1	1	1	1	−1	−1	−1
$\phi_{121}=\Phi_3$	1	1	−1	1	−1	−1	1	−1
$\phi_{112}=\Phi_4$	1	1	1	−1	−1	1	−1	−1
$\phi_{221}=\Phi_5$	1	−1	−1	1	−1	1	−1	1
$\phi_{212}=\Phi_6$	1	−1	1	−1	−1	−1	1	1
$\phi_{122}=\Phi_7$	1	1	−1	−1	1	−1	−1	1
$\phi_{222}=\Phi_8$	1	−1	−1	−1	1	1	1	−1

Table 4: IIR's (characters) table of $G_{\hat{\Theta}}(\hat{A}_\mu)$.

Table 5: IIR's table of $G_{\Theta}^{(2)}(\psi)$.

IIR's $G_{\Theta}^{(2)}(\psi)$	I	C	$-I$	$-C$	P	CP	$-P$	$-CP$	T	CT	$-T$	$-CT$	PT	Θ	$-PT$	$-\Theta$
φ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
φ_2	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1
φ_3	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
φ_4	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
φ_5	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1
φ_6	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1
φ_7	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1
φ_8	1	$-$	1	$-$	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
φ_9	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}$	$\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}$	$\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}-i & 0\\ 0 & -i\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
φ_{10}	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$	$\begin{pmatrix}-i & 0\\ 0 & -i\end{pmatrix}$	$\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}$	$\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

Ch $G_{\Theta}^{(2)}(\psi)$	$[I]$	$[-I]$	$2[C]$	$[T]$	$[-T]$	$2[P]$	$2[CP]$	$2[CT]$	$2[PT]$	$2[\Theta]$
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	-1	-1	1	-1	1	-1	1
χ_3	1	1	-1	1	1	1	-1	-1	1	-1
χ_4	1	1	1	-1	-1	1	1	-1	-1	-1
χ_5	1	1	-1	-1	-1	-1	1	1	1	-1
χ_6	1	1	-1	1	1	-1	1	-1	-1	1
χ_7	1	1	1	-1	-1	-1	-1	-1	1	1
χ_8	1	1	1	1	1	-1	-1	1	-1	-1
χ_9	2	-2	0	$2i$	$-2i$	0	0	0	0	0
χ_{10}	2	-2	0	$-2i$	$2i$	0	0	0	0	0

Table 6: Characters table of $G_{\Theta}^{(2)}(\psi)$.